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Integrals of motion for three-dimensional non-Hamiltonian dynamical systems

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Abstract. In this paper, the problem of finding integrals of motion of three-dimensional dynamical systems is analysed. We introduce a new type of direct method in the search of parameter values for which an integral of motion exists. This method consists in proposing an ansatz for the integral that explicitly shows the dependence with respect to one of the phase space coordinates of the system. We apply this procedure to the reduced three-wave interaction problem and to the Rabinovich system. For both models new integrals of motion are found.

1. Introduction

Dynamical systems described by coupled nonlinear ordinary differential equations are a common occurrence in many branches of applied science. These equations are generally non-Hamiltonian (i.e. not derived from a Hamiltonian function) and describe the time evolution of physical processes which are mainly dissipative in character. During such processes phase space volumes contract and the motion is often attracted by fixed points or periodic orbits.

However, the great majority of nonlinear systems have ranges of parameter values for which the solutions approach a much more complicated type of attractor. These are subsets of phase space with a Cantor-like structure, called strange attractors, on which the motion is widely chaotic in the sense that it depends sensitively on the choice of initial conditions.

The subject of self-generated chaotic behaviour in simple dynamical systems is by now widely recognized as one of the most interesting and intensively studied areas of mathematical physics [1, 2].

The fact that most deterministic systems of nonlinear mechanics are non-integrable and possess large classes of solutions with truly random properties is generally appreciated.

However, to date, important questions such as how to identify integrable dynamical systems, or how to determine the size of regions of chaotic behaviour in phase space, remain unresolved [3].

Of course, the main difficulty with integrating the ordinary differential equations of any dynamical system is that these equations are generally nonlinear and involve several degrees of freedom, which are coupled to each other in a non-trivial way. So far, most of the progress has been done in the area of integrable Hamiltonian systems, where a number of rigorous results are known [4]. However, despite the long history

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of the problem no general method is available to date, even for deciding whether a given dynamical system is integrable, let alone integrating its equations of motion explicitly. In recent years, a direct method has been proposed for identifying integrable dynamical systems by requiring that their solutions possess no movable (i.e. initial condition dependent) singularities other than poles in the complex time plane [5]. This so-called Painlevé property was originally adopted by Kowalevskaya in her celebrated integration of a special case of rigid body motion, and was employed by Painlevé and coworkers in their exhaustive studies of integrable equations of second order [6]. The Painlevé property has been used successfully to identify new integrable Hamiltonian systems, as well as integrable cases of non-Hamiltonian systems such as the Lorenz equations, the Lotka–Volterra system, etc. [5].

Unfortunately, this method, of high practical value, is not supported by a firmly established mathematical basis. Moreover, the Painlevé analysis method puts emphasis on complex analytic integrals and is not well adapted to the search for integrals in the real domain.

Very recently another method for finding integrals for three-dimensional ordinary differential equations has been employed [7, 8]; the method is based on the Frobenius integrability theorem. The main point of the method is to detect the values of the parameters for which the system can have first integrals which at the same time are integrals of some non-trivial linear system of three differential equations with constant coefficients. With this procedure new constants of motion have been found for the Lotka–Volterra systems. The Lie symmetry method has also been applied for searching for constants of motion of dynamical systems, but, in general, no new results, when compared with other procedures have been found [9].

Another usual procedure is to make a specific ansatz for the integral, as for example to propose a polynomial of a given degree in the phase space coordinates of the system. This procedure has been employed by Kus [10] for obtaining new constants of motion for the Lorenz model which do not fulfil the Painlevé criterion.

In this paper we also employ an ansatz for the integral, but of a more general character. We propose a polynomial in one of the coordinates of the system, the coefficients of which are unknown functions of the other coordinates. These functions must satisfy an overdetermined set of partial differential equations, which are compatible only for particular values of the parameters of the system.

We employ this method to study the three-wave interaction problem, the Rabinovich system and the Lorenz model.

For the three-wave problem and the Rabinovich system we recover systematically all the integrals of motion previously known and we also find several new ones. In the case of the Lorenz model we recover all the known results obtained by other methods but we are not able to find new cases for which an integral of motion exists.

In section 2 we treat the three-wave interaction problem. In section 3 we analyse the Rabinovich system. The Lorenz model is briefly considered in section 4, while in section 5 some conclusions are established.

2. The reduced three-wave interaction problem

As a first three-dimensional dynamical system we will consider the reduced three-wave interaction problem. In this system, three quasisynchronous waves interact in a plasma with quadratic nonlinearities [11].

The dynamical evolution of this system is determined by the following equations:

$$\begin{aligned}\dot{x} &= \gamma x + \delta y + z - 2y^2\\ \dot{y} &= \gamma y - \delta x + 2xy\\ \dot{z} &= -2z - 2zx.\end{aligned} \tag{2.1}$$

Here x, y, z are proportional to the amplitudes of the three waves respectively, δ measures the detuning from synchronism, and the other linear terms describe effects of dissipation and the pumping of external energy [11]. The overdot indicates a derivative with respect to time t. This system has been analysed in [12] by means of the Painlevé method. The following cases for which an integral of motion exists have been found:

(i) $\gamma = 0$, δ arbitrary, with the integral

$$I = z(y - \delta/2)\,\mathrm{e}^{2t}. \tag{2.2}$$

(ii) $\gamma = -1$, δ arbitrary, with the integral

$$I = (x^2 + y^2 + z)\,\mathrm{e}^{2t}. \tag{2.3}$$

In the special case $\delta = 0$, a second integral has been obtained for case (ii):

$$I = zy\,\mathrm{e}^{3t} \tag{2.4}$$

(there are misprints in (3.7) and (3.11) of [12]; the correct expressions are given in (2.2) and (2.3) of the present paper).

When the system admits an integral of motion, as in cases (i) and (ii), the analysis of its dynamical behaviour, especially in the $t \to \infty$ limit, is greatly simplified, and a non-chaotic behaviour is observed in general. If we analyse the mathematical structure of the integrals (2.2), (2.3) and (2.4), we observe that in the three cases the integral is linear in the variable z. Are these integrals the only ones that are linear in z? We have analysed this problem and we have found two other constants of motion of this type. By the nature of the method employed to find these results, we can be sure that no other integrals linear in z can be found.

The general form of a constant of motion linear in z and with an exponential time dependence is

$$I = (a_1(x, y) + a_2(x, y)z)\,\mathrm{e}^{\alpha t} \tag{2.5}$$

where a_1 and a_2 are arbitrary functions of x and y and α is an arbitrary parameter.

Taking into account (2.1), the condition $\mathrm{d}I/\mathrm{d}t = 0$ leads to the following equation:

$$\begin{aligned}a_{2x}z^2 &+ [(\gamma x + \delta y - 2y^2)a_{2x} + a_{1x} + (\gamma y - \delta x + 2xy)a_{2y} + (\alpha - 2(1+x))a_2]z\\ &+ (\gamma x + \delta y - 2y^2)a_{1x} + (\gamma y - \delta x + 2xy)a_{1y} + \alpha a_1 = 0\end{aligned} \tag{2.6}$$

where the suffixes x and y indicate partial derivatives.

As (2.6) must hold for arbitrary values of z, we must impose the following conditions on the functions a_1 and a_2:

$$a_{2x} = 0 \tag{2.7a}$$

$$(\gamma x + \delta y - 2y^2)a_{2x} + a_{1x} + (\gamma y - \delta x + 2xy)a_{2y} + (\alpha - 2(1+x)a_2) = 0 \tag{2.7b}$$

$$(\gamma x + \delta y - 2y^2)a_{1x} + (\gamma y - \delta x + 2xy)a_{1y} + \alpha a_1 = 0. \tag{2.7c}$$

From (2.7a) we have $a_2 = a_2(y)$, while (2.7b) and (2.7c) become

$$a_{1x} = -(\gamma y - \delta x + 2xy)a_2' - (\alpha - 2(1+x)a_2) \tag{2.8a}$$

$$(\gamma x + \delta y - 2y^2)a_{1x} + (\gamma y - \delta x + 2xy)a_{1y} + \alpha a_1 = 0 \tag{2.8b}$$

where the prime indicates a derivative with respect to y.

We have two partial differential equations for only one function of the two variables x and y, i.e. the function a_1. The function a_2 of the variable y and the parameters α, δ and γ must be chosen in order to make the two equations (2.8) compatible.

By replacing (2.8a) in (2.8b) we obtain an expression for a_{1y} in terms of a_1, a_2, a_2', a_2'', x and y, as follows:

$$a_{1y} = -\frac{\alpha a_1 + (2+2x-\alpha)(\gamma x + \delta y - 2y^2)a_2}{\gamma y - \delta x + 2xy} + (\gamma x + \delta y - 2y^2)a_2'. \tag{2.9}$$

After imposing the integrability condition $a_{1xy} = a_{1yx}$ between (2.8a) and (2.9) we obtain, after some calculations, an explicit expression for a_1, as follows:

$$\begin{aligned} a_1 = \frac{1}{\alpha(2y-\delta)} \{ & [(2\alpha + 2\alpha x - \alpha^2)(\gamma y - \delta x + 2xy) \\ & - 4(\gamma - 2 + \alpha)y^3 - 2\delta(-2\alpha + 4 - \gamma)y^2 \\ & + (2-\alpha)(\delta^2 + \gamma^2)y - 2\delta\gamma x^2 + 4\gamma x^2 y + 4\gamma^2 xy]a_2 \\ & + 2(\gamma y - \delta x + 2xy)^2(1 - \gamma - \alpha)a_2' - (\gamma y - \delta x + 2xy)^3 a_2'' \} \end{aligned} \tag{2.10}$$

where we have assumed $\alpha \neq 0$.

The case $\alpha = 0$ has also been analysed, but it does not lead to new results. From expression (2.10) we can obtain a_{1x} and a_{1y}. Then, we replace the expressions of a_{1x}, a_{1y} and a_1 in (2.8a). After a long calculation, and taking into account that a_2 is a function only of y, we must impose the following conditions, in order to satisfy (2.8a) identically for arbitrary values of x:

$$a_2'' = 0 \tag{2.11}$$

$$[4\gamma\delta + 2\alpha\delta - (8\gamma + 4\alpha)y]a_2 + (2y - \delta)^2(3\alpha + 4\gamma - 4)a_2' = 0 \tag{2.12}$$

$$2\gamma(2\gamma + \alpha)a_2 + \gamma(2y - \delta)[-3\alpha + 4(1-\gamma)]a_2' = 0. \tag{2.13}$$

From (2.11) we have

$$a_2(y) = C_1 y + C_2 \tag{2.14}$$

where C_1 and C_2 are arbitrary constants.

After replacing (2.14) in (2.12) and (2.13), we obtain the following algebraic equations for the parameters α, δ, γ, C_1 and C_2:

$$C_1(\gamma + \alpha - 2) = 0 \tag{2.15a}$$

$$(5\alpha\delta + 6\delta\gamma - 8\delta)C_1 + 2(\alpha + 2\gamma)C_2 = 0 \tag{2.15b}$$

$$\gamma(\gamma - 2 + \alpha)C_1 = 0 \tag{2.15c}$$

$$\delta[(3\alpha\delta + 4\delta\gamma - 4\delta)C_1 + 2(\alpha + 2\gamma)C_2] = 0 \tag{2.15d}$$

$$\gamma[(3\alpha\delta + 4\delta\gamma - 4\delta)C_1 + 2(\alpha + 2\gamma)C_2] = 0. \tag{2.15e}$$

There are four cases to consider:

(i) $C_1 = C_2 = 0$. After using (2.8b) it can be shown that no constant of motion is obtained.

(ii) $C_1=0$, $C_2\neq 0$. In this case, taking $\alpha=-2\gamma$, (2.15) are satisfied; using (2.8*b*) we found that the only possibility to obtain a non-trivial result is to take $\gamma=-1$. Therefore, we have $C_1=0$, $\alpha=2$ and δ arbitrary (it can be taken that $C_2=1$ without loss of generality). The resulting constant of motion is given in (2.3) and was found by Bountis *et al* in [12] by using the Painlevé approach.

(iii) $C_1\neq 0$, $C_2\neq 0$. In this case if we take $\alpha=2-\gamma$ and $C_2=-\delta C_1/2$, (2.15) are satisfied. It can be shown that this is the only solution with $C_1\neq 0$. A further calculation shows that in order to satisfy (2.8*b*) we must have $\gamma=0$ or $\delta=0$. For the case $\gamma=0$ we obtain $\alpha=2$, $a_1=0$ and $a_2=(2y-\delta)/2$ (we can take $C_1=1$ without loss of generality). The resulting integral is the expression (2.2), also found by Bountis *et al.*

If we consider the second possibility $\delta=0$, we obtain $\alpha=2-\gamma$, $a_1=0$ and $a_2=y$ ($C_1\equiv 1$). The constant of motion is

$$I=yz\,\mathrm{e}^{(2-\gamma)t} \tag{2.16}$$

with γ arbitrary and $\delta=0$. This result has not been found by Bountis *et al* and is new to our knowledge.

(iv) $C_1\neq 0$, $C_2=0$. The only solution of (2.15) is $\gamma=-2$, $\alpha=4$, and (2.8*b*) is identically satisfied in this case. Therefore, the resulting constant of motion is

$$I=(y^2+x^2+2yz/\delta)\,\mathrm{e}^{4t} \tag{2.17}$$

with δ arbitrary, which also constitutes a new result for the three-wave problem.

We have also employed a more general ansatz than (2.5) given by

$$I=z^{\beta}[a_1(x,y)+a_2(x,y)z] \tag{2.18}$$

where β is an arbitrary parameter. The form of the ansatz (2.18) is motivated by the fact that the third of equations (2.1) is homogeneous to the first degree in the variable z. The resulting equations for the functions a_1 and a_2 are slight modifications of (2.7), with an additional parameter β at our disposal. However, after long calculations, we have arrived at the conclusion that no new integrals of motion of the form (2.18) exist for $\beta\neq 0$.

3. The Rabinovich system

The Rabinovich system is a three-wave interaction model whose dynamical evolution is determined by the following equations:

$$\begin{aligned}\dot{x}&=hy-\nu_1x+yz\\ \dot{y}&=hx-\nu_2y-xz\\ \dot{z}&=-\nu_3z+xy\end{aligned} \tag{3.1}$$

where ν_1, ν_2, ν_3 are the damping rates and h is proportional to the driving amplitude of the feeder wave [11].

This model has been studied by Bountis *et al* by means of the Painlevé method. The following cases for which a constant of motion exists have been found by these authors:

$$I=(x^2+y^2-4hz)\,\mathrm{e}^{2\nu t} \tag{3.2}$$

with the conditions $\nu_1=\nu_2=\nu>0$, $\nu_3=2\nu$;

$$I=(x^2-y^2-2z^2)\,\mathrm{e}^{2\nu t} \tag{3.3}$$

with the conditions $\nu_1=\nu_2=\nu_3=\nu>0$; and

$$I=(x^2+y^2)\,e^{2\nu t} \tag{3.4}$$

with $h=0$, $\nu_1=\nu_2=\nu>0$.

The three integrals of motion presented above are linear in x^2. Guided by this fact we propose the following ansatz in order to find new integrals of this type:

$$I=[a_1(y,z)+a_2(y,z)x^2]\,e^{\alpha t} \tag{3.5}$$

where a_1 and a_2 are functions only of y and z, and α is an arbitrary parameter.

Obviously, other forms for the ansatz can be proposed. For instance, constants of motion that are linear in y^2 or quadratic in z can be analysed. Also, polynomials in x^2 or y^2 of degree greater than one can be considered. In this paper we have limited our study to the form (3.5). The condition $dI/dt=0$ leads to the following equations for the functions a_1 and a_2:

$$\nu_2 y a_{1y}+\nu_3 z a_{1z}=\alpha a_1 \tag{3.6a}$$

$$(h-z)a_{1y}+ya_{1z}+2y(h+z)a_2=0 \tag{3.6b}$$

$$\nu_2 y a_{2y}+\nu_3 z a_{2z}=(\alpha-2\nu_1)a_2 \tag{3.6c}$$

$$(h-z)a_{2y}+ya_{2z}=0. \tag{3.6d}$$

From (3.6c) and (3.6d) we determine a_{2x} and a_{2y} in terms of a_2, y and z. By imposing the compatibility condition $a_{2xy}=a_{2yx}$ the function a_2 can be found.

After this we replace the expression for a_2 and its partial derivatives in (3.6c) and (3.6d). In order to satisfy these equations identically we must impose some algebraic conditions on the parameters of the model.

The second step of the calculation consists of replacing the a_2 expression in (3.6b) and to repeat the procedure for (3.6c) and (3.6d) with (3.6a) and (3.6b).

After some calculations we have found by means of the procedure described above the following new constants for this system (obviously, we have also reobtained the known cases (3.2), (3.3) and (3.4)):

$$I=y^2+(h-z)^2 \tag{3.7}$$

with $\nu_2=\nu_3=0$, h and ν_1 arbitrary;

$$I=x^2-(z+h)^2 \tag{3.8}$$

with $\nu_1=\nu_3=0$, h and ν_2 arbitrary;

$$I=(y^2+z^2)\,e^{2\nu_3 t} \tag{3.9}$$

with $\nu_2=\nu_3$, $h=0$, ν_1 and ν_3 arbitrary;

$$I=(x^2-z^2)\,e^{2\nu_3 t} \tag{3.10}$$

with $\nu_1=\nu_3$, $h=0$, ν_2 and ν_3 arbitrary.

Following the terminology of Bountis *et al* [12], we call a case partially integrable when a constant of motion exists. In general, two independent integrals are necessary in order to have complete integrability of the equations of motion in terms of known functions.

In this work we do not study the integrability problem. We only investigate the conditions for which at least one integral of motion can be found. The existence of such an integral rules out the possibility of chaotic behaviour, strange attractors, etc., but does not assure integrability.

4. The Lorenz model

The celebrated and intensively studied Lorenz equations

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= -y+\rho x - xz \\ \dot{z} &= -bz + xy \end{aligned} \tag{4.1}$$

arise in simple models of hydrodynamic turbulence [13].

Several authors have analysed the problem of finding particular cases of the values of parameters σ, ρ, b for which an integral of motion exists [7, 9, 10, 12, 14]. In that case, one of course would expect no strange attractors or chaotic behaviour to be present. All the integrals of motion that have been found for the Lorenz model are polynomials of degree $\leqslant 4$ in the variables x, y and z. In particular, with respect to z they are polynomials of degree $\leqslant 2$.

Taking into account this fact we have considered the following ansatz for I:

$$I = [a_1(x, y) + a_2(x, y)z + a_3(x, y)z^2]\, e^{\alpha t}. \tag{4.2}$$

The condition $dI/dt = 0$ leads to the following equations for the functions a_1, a_2 and a_3:

$$a_{3_y} = 0 \tag{4.3a}$$

$$\sigma(y-x)a_{3x} - xa_{2_y} + (\alpha - 2b)a_3 = 0 \tag{4.3b}$$

$$\sigma(y-x)a_{2x} - xa_{1_y} + (\rho x - y)a_{2_y} + 2xya_3 + (\alpha - b)a_2 = 0 \tag{4.3c}$$

$$\sigma(y-x)a_{1x} + (\rho x - y)a_{1_y} + xya_2 + \alpha a_1 = 0. \tag{4.3d}$$

We have solved this system of equations by the same procedure employed in sections 2 and 3.

After a long calculation we have reobtained all the known integrals of motion of the Lorenz model, but we have not found new results. In consequence, we can assert that the Lorenz model has no other polynomial constants of motion of degree $\leqslant 2$ in the variable z. A more general ansatz than (4.2) can be considered (i.e. a polynomial of degree 3 in z) but the calculations become much more involved.

5. Conclusions

We have employed in this paper a direct method for the search of constants of motion of three-dimensional dynamical systems. The method consists in proposing an ansatz for the invariant which is a polynomial of a given degree in one of the coordinates of the phase space of the system. The coefficients of this polynomial are arbitrary functions of the other coordinates of the phase space.

These functions must satisfy a system of partial differential equations. We have shown in the examples treated in this paper that this system of equations can be solved, but the complexity of the calculations strongly increases with the degree of the polynomial proposed in the ansatz.

For dissipative systems, this type of ansatz has not been employed before to our knowledge.

In general, the ansatz previously utilized before consists of proposing a polynomial in all the coordinates of the system, where the coefficients are free parameters to be determined in order to have a constant of motion, as for example in the work of Kus [10].

The method employed in this paper has proven to be a very efficient tool in the search for constants of motion for three-dimensional dynamical systems. We have found several new integrals for the three-wave interaction problem and for the Rabinovich model. For the Lorenz equations we have not been able to find new constants of motion but we have reobtained all the known ones.

The method is also applicable to higher-dimensional dynamical systems and we think that it represents a complement to the usual approach based on the Lie symmetry method, the Painlevé property and the Frobenius integrability theorem recently introduced by Strelcyn and Wojciechowski.

Note added. The referee has pointed out that the method employed in this paper is very similar to much work being done on the search for small-amplitude limit cycles and centre conditions for two-dimensional systems with polynomial right-hand sides. A recent review on problems can be found in [15].

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